STUDENT EXPERIENCES WITH VISUALIZATION OF ABSTRACT ENTITIES IN THE LEARNING OF MULTIVARIABLE CALCULUS

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ABSTRACT

This paper focuses on undergraduate students taking Calculus course in a BEd degree programme offered at Great Zimbabwe University. The paper analyses student conceptions and misconceptions on the graph of surfaces. The results indicated that students conceptualise better through use of visuals and also that technology plays a key role in enhancing better understanding of calculus concepts.

Keywords: Visualisation, technology and surfaces

INTRODUCTION:

Students have interacted with drawing graphs of functions as early as primary level. Graphs sometimes are very easy to conceptualise the characteristics of a model in problem solving than explanations. Practices in mathematics problem solving are often based on verbal representations that make use of logical connectives in sequential reasoning. Recent research in mathematics teaching (Diezmann, 1997) however has advocated the use diagrammatic explanation to assist comprehension. In this study the researcher explores students' conceptions and misconceptions on the graphs of surfaces by undergraduate students taking the course of Calculus of several variables. The objective of the research is to reinforce students 'conceptions of graphs on single variable Calculus , transferring such concepts to the graphs of surfaces. For example in single variable Calculus students were able to draw the sketch of y=sinx , being able evaluate its domain and range. Are students going to use the same analytic methods for the graph of z=sinxy ? What properties are preserved by such functions in space geometry? However the main question is: what conceptions and misconceptions of the graphs of surfaces do students develop in a course which emphasises visualization , uses of technology, and deemphasises symbolic manipulation?

THEORETICAL FRAMEWORK:

Stewart J.(1999) asserted that the primary aim of Calculus instruction is the emphasis on understanding concepts. In fact the impetus for the current Calculus reform movement came from the Tulane Conference in 1986, which formulated as their first recommendation : focus on conceptual understanding. He also went on to emphasise that the most important way to foster conceptual understanding is through the problems we assign. In this study the researcher also used the Cognitive Dissonance theory (L.Festinger,1957) and Cognitive Flexibility theory (R.Spiro et al,1990). Dissonance theory applies to all situations involving attitude formation and change. It is especially relevant to decision -making and problem -solving. When there is an inconsistency between attitudes (dissonance) something must change to eliminate the dissonance. This theory shall be useful to students when constructing graphs of functions in space . Some properties of functions of single variable are going to be inconsistent with functions of several variables, for example, the criteria for evaluating critical points. Flexibility theory is especially formulated to support the use of interactive technology. Its main principle is that learning activities must provide multiple representations of content. This will be applied when Computer Algebra Systems (Mathematica) is going to be used to draw graphs and level curves of functions of several variables. Students should then associate functions with their graphs and level curves creating a chain of symbolic representation, plane representation and space representation.

VISUALIZATION:

Is a picture worth a thousand words? It seems so ,as historical accounts of scientific discovery and invention have shown that visualisation is a powerful cognitive tool (Rieber,1995). The term visualisation is familiar to us from common usage and fundamentally means "to form and manipulate a mental image". In everyday life visualisation is essential to problem solving and spatial reasoning as it enables people to use concrete means to grapple with abstract images. The process of visualisation involves the formation of images, with paper and pencil or even mentally , to investigate , discover and understand concepts ,facts and ideas. Pictorial and visual forms of representation can offer advantages over text-based resources by offering scope for : displaying spatial interrelationships and facilitating perceptual inference (e.g .relative size of objects).

Visualisation has been accounted for by a number of theorists who have indicated its centrality in reasoning and learning. Bruner (1984, p. 99) characterises two alternative approaches to solving problems, one being intuitive, the other analytic. McLoughlin & Krakowski (2001) argue that visual representation of ideas is just as much a part of the learning process as using language and other symbolic representations, yet current theories of learning with technology do not always highlight this important dimension of the learning process. Theorists have emphasised that visual thinking is a fundamental and unique part of our perceptual processes and that visualisation is a partner to the verbal and symbolic ways we have of expressing ideas and thoughts.

GRAPHS OF SURFACES:

It is very difficult to visualise a function f of three variables by its graph, since that would lie in a four-dimensional space. However we do gain some insight into f by examining its level surfaces, which are the

surfaces with equations f(x,y,z)=k , where k is a constant. For example find the level surfaces of the function $f(x,y,z) = x^2 + y^2+z^2$.
Solution: The level surfaces are $x^2 + y^2+z^2 = k$, where k is positive . These form a family of concentric spheres with radius square root of k and centre (0,0,0).

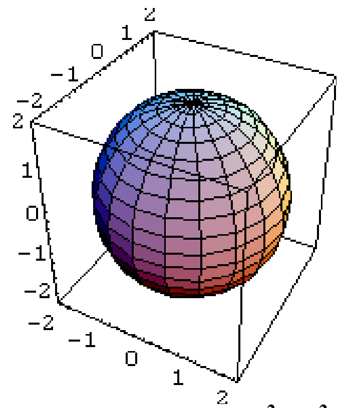


Fig.1.The graph of the function $x^2 + y^2+z^2=4$ (k=4)

The above graph can further be represented by level curves on a plane , for instance f(x,y) =c where c is a constant. In this case z=f(x,y) and z=c will result in $x^2 + y^2 = 4-c$ for c <4 .

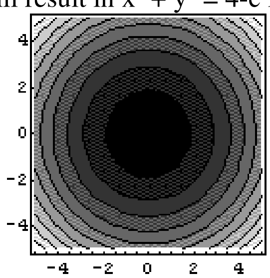


Fig2. Level curves for the function $f(x,y)=x^2+y^2$

TECHNOLOGY:

The availability of technology makes it more important to clearly understand the concepts that underlie the images on the screen. When properly used graphing calculators and computers are powerful tools for discovering and understanding those concepts. The two diagrams above were drawn using Mathematica Version 4.0. Technology does not make pencil and paper obsolete. Hand calculations and sketches are often preferable to technology for illustration and reinforcing some concepts. Both instructors and students need to develop the ability to decide where the hand or the machine is appropriate. Thomas ,D (1995) argues that contemporary graphing technologies in the hands of students will enhance their learning. Almost all Computer Algebra Systems are capable of drawing graphs in two and three dimensions that is why they are excellent visualisation tools. The software tools and the computer screen can serve as a *scaffold* or support for dialogue, reflection and learning, becoming in effect cognitive tools for learning.

METHODOLOGY:

DESIGN OF STUDY:

In this study the major objective is to identify students 'conceptions and misconceptions on the graphs of surfaces. The sample for this experiment is composed of 13 B.Ed. students specialising in mathematics at secondary level taken from the population of all B.Ed. students in the Department of Curriculum Studies at Great Zimbabwe University. The participants are first years taking a course of Advanced Calculus. The course covers both single variable and multivariable calculus. The design of the study is of the form

Experiment –Observe (X-O). The research methodology is quantitative in nature.

RESEARCH INSTRUMENT:

The research instrument is a calculus exercise taken from the topic of Geometry of Space. The exercise consists of two questions. The objectives of question 1 is to describe various given functions of surfaces and also to sketch the graph of the respective surface. The objective of question 2 is to match function, graph and level curve respectively.(See table 1 below)

Table 1. Research instrument (Test on students conceptions and misconceptions of surfaces)

1. Answer all questions on spaces provided.

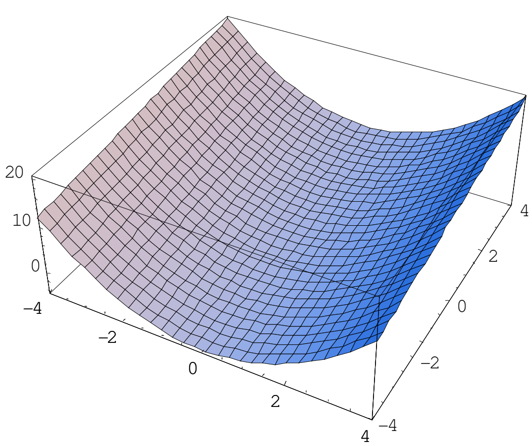
Function/equation/inequality	Describe the region corresponding to the given equation/inequality in R^3	Sketch the graph of the region
(a) X=9		
(b) X=-8		
(c) $X^2 + y^2 + z^2 = 1$		
(d) $1 < X^2 + y^2 + z^2 < 25$		
(e) X+y+z=1		

2.Match the function with its graph and level curves.

FUNCTION (a-e)	GRAPH (I-V)	LEVEL CURVE (A-E)
(a) z=sinxy		
(b) z=x+y		
(c) $z=x^2+y^2$		
(d) $z=x^2+y$		
(e) $z=x^2-y^2$		

Fig.3. Graphs of functions in question 2.

I.



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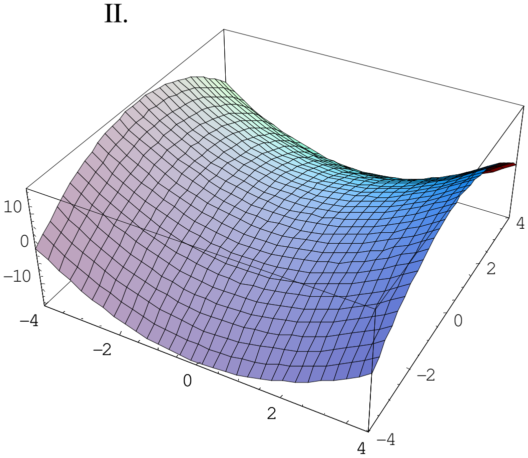

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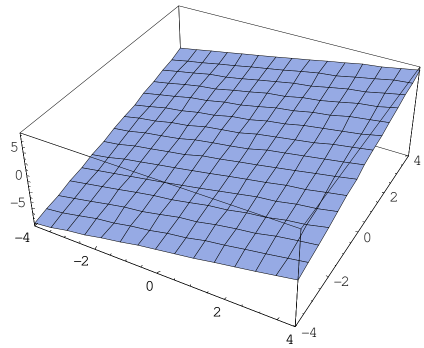

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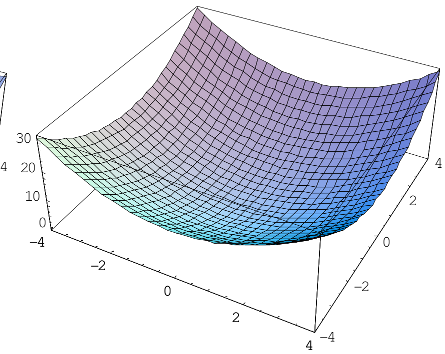

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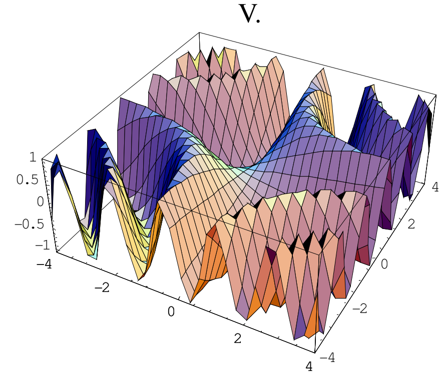
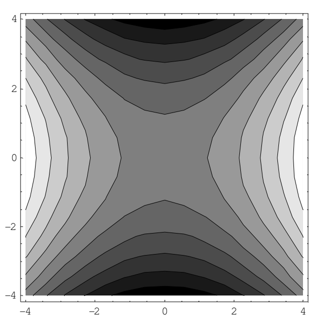


Fig.4. Level curves of functions in question 2.

A



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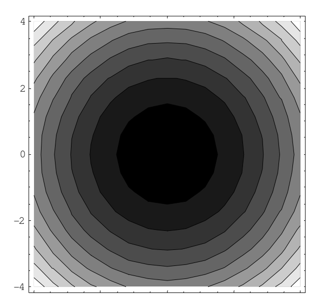

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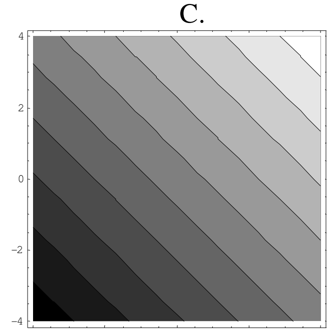

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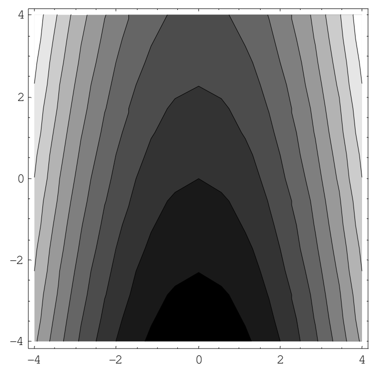

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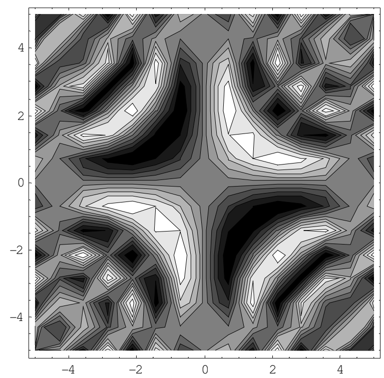

Each task was graded on a four point scale with the following scale: Correct (3) , Partially correct (2) , Incorrect (1), Missing (0). The test consisted of three factor levels which were to be analysed. The first being descriptions of regions (question1) , the second factor being sketching of the graphs of regions (question1) and the third factor was matching function -graph-level curve (question2).

TREATMENT:

The study was conducted in a course (Advanced Calculus) designed to learn the concepts of both Calculus 1 and Calculus2 in one semester. Students were first taught single variable calculus (Calculus1) and then multivariable calculus (Calculus 2). The research is then focussed on the first concepts of multivariable calculus which is the geometry of surfaces. This section is very vital in multivariable calculus. For students to be able to perform multiple integration it is vital that students are able to describe and sketch the graphs and level curves of multivariable functions. The concepts on Question1 were function and graphical representation in space using pencil and paper. The concepts on Question2 were function, graphical representation in space and level curves on the x-y plane. The graphs of functions and level curves were provided to the students drawn using Mathematica 4. Students were not tasked to use computers to draw these graphs.

STUDENTS' PROCEDURES AND CONCEPTIONS:

The analysis of students' written responses revealed significant information regarding the nature and characteristics of students ' conceptions on surfaces. The table below is a summary of results of the distribution of scores on the written exercise. [Correct (C)=3 , Partially Correct (PC)=2 ,Incorrect (I)=1, Missing (M)=0]. For convenience , in this research all Correct and Partially Correct answers shall be regarded as students' conceptions and Incorrect answers shall be classified as errors and / or misconceptions (depending on the nature of error).

Table2: Summary of results of the written exercise

	1(a)	1(b)	1(c)	1(d)	1(e)	2(a)	2(b)	2(c)	2(d)	2(e)	Total
A	2	2	2	2	3	3	3	3	3	3	C=6,PC=4,I=0,M=0
B	2	2	3	1	2	3	3	2	2	2	C=3,PC=6,I=1,M=0
C	2	2	3	2	3	3	3	3	3	3	C=7,PC=3,I=0,M=0
D	2	2	3	3	2	2	3	2	1	2	C=3,PC=6,I=1,M=0
E	3	3	3	2	3	3	3	2	2	3	C=7,PC=3,I=0,M=0
F	3	3	3	2	1	3	3	2	2	2	C=5,PC=4,I=1,M=0
G	3	3	1	3	3	2	3	2	2	1	C=5,PC=3,I=2,M=0
H	2	2	1	1	1	1	3	3	1	2	C=2,PC=3,I=5,M=0
I	1	1	1	1	2	2	3	2	3	2	C=2,PC=4,I=4,M=0
J	2	2	3	2	2	3	3	3	2	2	C=4,PC=6,I=0,M=0
K	2	2	2	1	2	2	1	3	1	2	C=1,PC=6,I=3,M=0
L	3	3	3	2	3	3	3	1	2	2	C=6,PC=3,I=1,M=0
M	2	2	2	1	3	3	3	2	2	2	C=3,PC=6,I=1,M=0
Total	**C=4 PC=8 I=1 M=0**	**C=4 PC=8 I=1 M=0**	**C=7 PC=3 I=3 M=0**	**C=2 PC=6 I=5 M=0**	**C=6 PC=5 I=2 M=0**	**C=8 PC=4 I=1 M=0**	**C=12 PC=0 I=1 M=0**	**C=5 PC=7 I=1 M=0**	**C=3 PC=7 I=3 M=0**	**C=3 PC=9 I=1 M=0**	**C=54,PC=57,I=19**

Table3: Classification and distribution of errors for each task

No.	Equation/Inequality	Errors/conceptions	Description
1(a)	X=9	General conception: It is a plane which cuts the x-axis at (9,0,0) Misconception: It is a line on the plane. It is Along the x-axis passing through (9,0) Misconception: It is a plane parallel to the point X=9.	Correct response: Plane parallel to the yz plane passing through point (9,0,0) Most students left out the concept of parallelism . Two students misconceive the plane for a point in the real line. One student misconceived a plane being parallel to a point. Students were able to draw the graph of the equation.
1(b)	Z=-8	Same as above 1(a)	Same as above 1(a)
1(c)	$X^2 + y^2 + z^2 = 1$	Misconception: It is a circle of radius 1 unit.	Correct response: Sphere with centre (0,0,0) and radius 1 unit. Two students mistook the equation of a circle on a plane with the equation of a sphere in space. A majority of students were not able to construct the graph of the relation.
1(d)	$1 < X^2 + y^2 + z^2 < 25$	Misconception: Two circles/discs General error: Students were not able to clearly Construct the required region.	Correct response: Region between two spheres of radii 1 unit and 5 units respectively. Almost al students were not able to describe the inequality and were not able to construct the required region.
1(e)	X+y+z=1	Misconception: It is a line with domain +1 and -1 Misconception: It is a prism triangular in shape…	Correct response: Plane cutting all the axes at points (1,0,0) , (0,1,0) and (0,0,1). Two students mistook this plane for a straight line and one student mistook the plane for a triangular prism. Students were able to construct the graph of the equation.
2(a)	z=sinxy	Not applicable	To answer this question students were to exploit the characteristics of sinx. Two students mistook the level curves of equation for that of $z=x^2+y$
2(b)	z=x+y	Not applicable	Only one student was not able to match both the graph and the level curves.
2(c)	$z=x^2+y^2$	Not applicable	A majority of students were not able to match the graph.
2(d)	$z=x^2+y$	Not applicable	A majority of students were not able to match the graph.
2(e)	$z=x^2-y^2$	Not applicable	A majority of students were not able to match the graph.

CONCLUSIONS AND RECOMMENDATIONS:

Taking consideration of the results from the given exercise , the researcher noted the following important observations which are answers to the main research question:

VISUALIZATION:

Question 1 comprised of two sections. The first section was description of the equation , without emphasis on visualization. The second section was drawing the graph of the given equation. The results showed that

about 85% of the students were more comfortable with drawing the graph rather than giving the description of the equation. Most students attempted section two first and then section one last by describing what they had constructed. It can be concluded that students conceptualise better through use of visuals.
Question 2 also comprised of two sections. However both sections were to be answered from the given visuals. Students were supposed to match the graph and its level curves. From the results the total for incorrect responses was 7 ,implying that 90% of the responses were either correct or partially correct. It can also be concluded that students performed much better on question2 than on question1.

USE OF TECHNOLOGY:

From the results of the exercise most students struggled with constructing of graphs of functions in 3-dimensions. Question 1 could have been enhanced if students were tasked to use the computer to construct these graphs as evidenced by the graphs provided in question2. Without use of technology clearly it could have been impossible to accomplish question2. This approach of questioning borrowed from the textbook of Stewart J(1999, pp 917-921) is only possible through the assistance of Computer Algebra Systems.
The students responded positively to question 2 , implying that the idea of using computers is very vital in calculus. The only drawback is that there is not enough time for both computers and calculus. It was observed that the use of computers had a dual purpose of facilitating and deepening the understanding of calculus concepts and also produced positive changes of students' attitudes toward the subject.

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